The Myrion Number System:
A Commutative Non-Associative Algebra
with a Singular Division Locus

Andrew Williams
Independent Researcher, United Kingdom

September 2026

Abstract

This paper defines a family of finite-dimensional real algebras, termed *Myrion algebras*, that extends the complex numbers through an ordered sequence of basis elements. For $p \geq 2$, the square of e_p descends to e_{p-1}, while $e_1^2 = -e_0$; products of distinct non-scalar basis elements follow the same descending-axis rule. Each finite-dimensional member is closed under its defining operations, unital, commutative, and distributive, and the two-dimensional member is isomorphic to the complex numbers. Every member of dimension greater than two is non-associative.

For an element x, multiplication defines a real-linear operator $L_x : y \mapsto x \star y$. The set

$$\Sigma_n = \{x \in \mathbb{M}_n : \det L_x = 0\}$$

is introduced as the *singular division locus*. For each fixed $\mathbb{M}_n$, an element outside this locus defines a bijective multiplication operator, and division by that element is the linear operation L_x^{-1}. The unique solution of $x \star r = 1$ is a reciprocal solution, but multiplication by it does not in general implement division. In dimension two the singular division locus reduces to the single point 0, recovering the familiar exceptional element of complex division. In higher dimensions it becomes a homogeneous algebraic set through the origin. The paper further proves that, for every finite $n \geq 2$, the squaring map $S_n(x) = x \star x$ is surjective, so every element of $\mathbb{M}_n$ has at least one square root within the same $\mathbb{M}_n$. Enlargement of the ambient Myrion algebra may nevertheless reveal additional roots; any finite-support square root can extend at most one basis level above the highest occupied level of the element being rooted. The paper derives coordinate multiplication formulae, proves the principal algebraic properties, gives explicit low-dimensional multiplication operators, and discusses the possible dependence of values on association trees. Possible relevance to structured neural computation is outlined, particularly where fixed or learned association trees and distance from the singular division locus may provide useful inductive biases. No empirical machine-learning claims are made.

Keywords: non-associative algebra; commutative algebra; hypercomplex numbers; square-root closure; singular division locus; division; structured neural networks; algebra-valued computation

1 Introduction

Complex-valued, quaternion-valued, and more general hypercomplex models have been studied as structured alternatives to real-valued computation. Their multiplication laws couple groups of real components and may provide useful parameter sharing or inductive bias in neural architectures

[6, 4, 8, 3, 1]. The best-known higher-dimensional constructions, including quaternions and octonions, trade familiar algebraic properties as dimension increases; octonions, for example, are non-associative [2, 5].

This paper introduces a different recursively ordered construction. The proposed Myrion algebras[1] are commutative but become non-associative above two dimensions. Their defining multiplication is intentionally simple: every basis element e_p with $p \geq 2$ squares to the preceding basis element, while $e_1^2 = -e_0$; a product of two non-scalar basis elements descends according to the lower-indexed factor.

The main contributions are:

1. a precise definition of the n-dimensional Myrion algebra $\mathbb{M}_n$;
2. proof that each finite-dimensional Myrion algebra $\mathbb{M}_n$ is closed under addition, real scalar multiplication, and Myrion multiplication;
3. explicit general and five-dimensional multiplication formulae;
4. proofs of commutativity, distributivity, non-associativity, and failure of power-associativity for $n > 2$;
5. proof that the squaring map $S_n(x) = x \star x$ is surjective for every finite $n \geq 2$, together with a bound on the dimensional extension of additional square roots;
6. a linear-operator treatment of reciprocal solutions and division;
7. definition of the singular division locus Σ_n;
8. explicit analysis of the complex and three-dimensional cases;
9. a cautious account of possible applications to structured neural computation.

The term *number system* is used descriptively. Algebraically, $\mathbb{M}_n$ is a finite-dimensional unital commutative real algebra. For $n > 2$ it is non-associative and has non-zero zero divisors; for example, $i \star (j - i) = 0$. It is therefore not a division algebra in the standard non-associative sense, which requires unique solutions of the division equations for every non-zero element.

2 Definition of the Myrion algebras

Definition 2.1 (Basis)**.** *For $n \geq 2$, let*

$$\mathbb{M}_n = \operatorname{span}_{\mathbb{R}}\{e_0, e_1, \ldots, e_{n-1}\},$$

where

$$e_0 = 1, \qquad e_1 = i, \qquad e_2 = j, \qquad e_3 = k, \qquad e_4 = l, \ldots$$

and e_0 is the multiplicative identity.

Definition 2.2 (Finite-support union)**.** *The compatible finite-support Myrion algebra is*

$$\mathbb{M}_{\text{fin}} = \bigcup_{n \geq 2} \mathbb{M}_n.$$

Its elements are precisely the finite real linear combinations of the basis elements. Statements below concerning finite-support elements are understood in this union; no claim is made for infinite series.

Definition 2.3 (Basis multiplication)**.** *The bilinear product $\star$ is defined on basis elements by*

$$e_0 \star e_p = e_p \star e_0 = e_p$$

[1] The name *Myrion* is formed from *myriad*, in its modern sense of "countless", together with the ending *-ion* found in the names of multidimensional number systems such as quaternions and octonions. It reflects that Myrion algebras are defined in every finite dimension, with no fixed upper dimensional limit.

and, for $p, q \geq 1$,

$$e_p \star e_q = \begin{cases} -e_0, & \min(p,q) = 1, \\ e_{\min(p,q)-1}, & \min(p,q) \geq 2. \end{cases}$$

The product of arbitrary elements is obtained by real-bilinear extension.

Thus

$$i^2 = -1, \qquad j^2 = i, \qquad k^2 = j, \qquad l^2 = k,$$

and

$$ij = -1, \qquad ik = -1, \qquad jk = i, \qquad jl = i, \qquad kl = j.$$

An arbitrary element is written

$$x = a_0 + a_1 i + a_2 j + \cdots + a_{n-1} e_{n-1}, \qquad a_r \in \mathbb{R}.$$

Proposition 2.4 (Closure of finite-dimensional Myrion algebras)**.** *For every finite $n \geq 2$, the space $\mathbb{M}_n$ is closed under addition, real scalar multiplication, and Myrion multiplication. Consequently, each $\mathbb{M}_n$ is a finite-dimensional unital real algebra in its own right.*

Proof. Let

$$x = \sum_{p=0}^{n-1} a_p e_p, \qquad y = \sum_{p=0}^{n-1} b_p e_p$$

be elements of $\mathbb{M}_n$. Componentwise addition gives

$$x + y = \sum_{p=0}^{n-1} (a_p + b_p) e_p \in \mathbb{M}_n,$$

and for every $\lambda \in \mathbb{R}$,

$$\lambda x = \sum_{p=0}^{n-1} (\lambda a_p) e_p \in \mathbb{M}_n.$$

For multiplication, bilinearity gives

$$x \star y = \sum_{p=0}^{n-1} \sum_{q=0}^{n-1} a_p b_q (e_p \star e_q).$$

If either basis factor is e_0, its product is the other factor and therefore remains in $\mathbb{M}_n$. If $p, q \geq 1$, then the defining rule gives either $-e_0$ or $e_{\min(p,q)-1}$. Since

$$0 \leq \min(p,q) - 1 \leq n - 2,$$

every basis product lies in $\mathrm{span}_{\mathbb{R}}\{e_0, \ldots, e_{n-1}\}$. Hence $x \star y \in \mathbb{M}_n$. $\square$

Remark 2.5. *The closure is downward with respect to the ordered basis: multiplication never creates a basis element of index greater than either non-scalar input index. A finite-dimensional implementation therefore requires no hidden higher-dimensional components.*

2.1 Cyclic behaviour of basis elements

For each non-scalar basis element e_p with $p \geq 1$, repeated left-associated multiplication gives

$$e_p,\ e_{p-1}, \ldots, e_1,\ -1,\ -e_p,\ -e_{p-1}, \ldots, -e_1,\ 1,$$

and then repeats. More precisely, if $u_1 = e_p$ and

$$u_{r+1} = u_r \star e_p,$$

then this sequence has period $2(p+1)$.

For example,

$$i,\ -1,\ -i,\ 1, \ldots$$

and

$$j,\ i,\ -1,\ -j,\ -i,\ 1, \ldots.$$

This cyclic description motivates the construction; the indexed multiplication rule is the formal definition.

3 Coordinate multiplication

Let

$$x = a_0 + \sum_{p=1}^{n-1} a_p e_p, \qquad y = b_0 + \sum_{q=1}^{n-1} b_q e_q,$$

and write

$$x \star y = \sum_{t=0}^{n-1} c_t e_t.$$

Proposition 3.1 (General coordinate formula)**.** *The scalar coefficient is*

$$c_0 = a_0 b_0 - a_1 \sum_{q=1}^{n-1} b_q - b_1 \sum_{p=2}^{n-1} a_p.$$

For $1 \leq t \leq n-2$*,*

$$c_t = a_0 b_t + a_t b_0 + a_{t+1} \sum_{q=t+1}^{n-1} b_q + b_{t+1} \sum_{p=t+2}^{n-1} a_p,$$

and

$$c_{n-1} = a_0 b_{n-1} + a_{n-1} b_0.$$

Proof. Expand

$$x \star y = \sum_{p,q} a_p b_q (e_p \star e_q)$$

and collect every pair (p, q) whose basis product equals e_t. For $t = 0$, the negative contributions arise whenever the minimum non-scalar index is 1. For $t \geq 1$, the non-scalar product contributes to e_t precisely when the minimum index is $t+1$. The identity terms contribute $a_0 b_t + a_t b_0$. $\square$

3.1 The five-dimensional product

For

$$x = a + bi + cj + dk + el, \qquad y = A + Bi + Cj + Dk + El,$$

the product is

$$\begin{aligned} x \star y = {} & [aA - b(B + C + D + E) - B(c + d + e)] \\ & + [aB + bA + c(C + D + E) + C(d + e)]i \\ & + [aC + cA + d(D + E) + De]j \\ & + [aD + dA + eE]k + [aE + eA]l. \end{aligned} \tag{1}$$

Squaring gives

$$\begin{aligned} x^2 = {} & [a^2 - b^2 - 2b(c + d + e)] \\ & + [2ab + c^2 + 2c(d + e)]i \\ & + [2ac + d^2 + 2de]j \\ & + [2ad + e^2]k + 2ae\,l. \end{aligned} \tag{2}$$

4 Basic algebraic properties

Theorem 4.1 (Commutativity)**.** *For every $n \geq 2$ and all $x, y \in \mathbb{M}_n$,*

$$x \star y = y \star x.$$

Proof. The basis rule depends only on $\min(p, q)$ and is therefore symmetric in p and q. Bilinear extension preserves this symmetry. □

Theorem 4.2 (Distributivity)**.** *For all $x, y, z \in \mathbb{M}_n$,*

$$x \star (y + z) = x \star y + x \star z, \qquad (x + y) \star z = x \star z + y \star z.$$

Proof. This follows immediately from the real-bilinear definition of $\star$. □

Proposition 4.3 (Flexibility)**.** *For every $n \geq 2$ and all $x, y \in \mathbb{M}_n$,*

$$(x \star y) \star x = x \star (y \star x).$$

Proof. By commutativity of the outer product,

$$(x \star y) \star x = x \star (x \star y).$$

By commutativity of the inner product, $x \star y = y \star x$, which gives the required identity. No associativity is used. □

Theorem 4.4 (Non-associativity above dimension two)**.** *Every $\mathbb{M}_n$ with $n > 2$ is non-associative.*

Proof. The subspace spanned by $\{1, i, j\}$ is contained in every $\mathbb{M}_n$ with $n > 2$. Within it,

$$i \star (j \star j) = i \star i = -1,$$

whereas

$$(i \star j) \star j = (-1) \star j = -j.$$

Since $-1 \neq -j$, associativity fails. □

Remark 4.5. *Myrion multiplication is commutative, so it does not preserve the order of the two operands in one binary product. For three or more factors, an association tree is nevertheless part of the syntactic expression and can affect its value. The output does not in general recover the tree uniquely: distinct trees may coincide for particular inputs or by identities such as flexibility.*

4.1 Non-associativity as a structural property

In the present construction, non-associativity should not be interpreted only as the loss of a familiar arithmetic law. It also creates additional structure. For three or more factors, the association tree can affect the value of the computation:

$$(x \star y) \star z \quad \text{and} \quad x \star (y \star z)$$

need not agree.

This makes it possible for a Myrion expression to distinguish alternative hierarchical compositions while retaining commutativity at each binary product. In computational applications this may be useful rather than defective: a model can, in principle, assign significance to how intermediate representations are grouped. The possible neural-network applications discussed later in the paper rely on this feature, and should therefore be viewed as exploiting non-associativity rather than merely tolerating it.

Proposition 4.6 (Failure of power-associativity)**.** *Every $\mathbb{M}_n$ with $n > 2$ is not power-associative.*

Proof. The element $j = e_2$ belongs to every $\mathbb{M}_n$ with $n > 2$. Since

$$j^2 = i, \qquad i \star i = -1, \qquad i \star j = -1,$$

the balanced product of four copies of j is

$$(j \star j) \star (j \star j) = i \star i = -1,$$

whereas the left-associated product is

$$((j \star j) \star j) \star j = (i \star j) \star j = (-1) \star j = -j.$$

Thus different parenthesisations of four copies of the same element give different results. Hence powers of degree four and above cannot in general be represented without specifying an association convention or tree. □

Proposition 4.7 (Failure of alternativity and the Jordan identity)**.** *For every $n > 2$, $\mathbb{M}_n$ is neither alternative nor a Jordan algebra.*

Proof. For alternativity, the element j gives

$$(j \star j) \star i = -1, \qquad j \star (j \star i) = -j,$$

so the left alternative law fails. The Jordan identity

$$(x^2 \star y) \star x = x^2 \star (y \star x)$$

fails on taking $x = y = j$, since

$$(j^2 \star j) \star j = -j \qquad \text{whereas} \qquad j^2 \star (j \star j) = -1.$$

□

Proposition 4.8 (Nonexistence of nontrivial multiplicative seminorms)**.** *For every $n > 2$, there is no non-zero real seminorm $N : \mathbb{M}_n \to [0, \infty)$ satisfying*

$$N(x \star y) = N(x)N(y)$$

for all $x, y \in \mathbb{M}_n$.

Proof. If N is not identically zero, then $N(1) = 1$, since $N(1) = N(1)^2$ and $N(x) = N(1 \star x) = N(1)N(x)$. From $i^2 = -1$, real homogeneity gives

$$N(i)^2 = N(-1) = 1,$$

so $N(i) = 1$. The zero-divisor identity $i \star (j - i) = 0$ then gives $N(j - i) = 0$. Since

$$(j - i)^2 = 1 + i,$$

we have $N(1 + i) = 0$. But

$$(1 + i) \star (1 - i) = 2,$$

and therefore

$$2 = N(2) = N(1 + i)N(1 - i) = 0,$$

a contradiction. □

5 Reduction to the complex numbers

The two-dimensional member

$$\mathbb{M}_2 = \mathrm{span}_{\mathbb{R}}\{1, i\}$$

has

$$i^2 = -1.$$

For $x = a + bi$ and $y = A + Bi$,

$$x \star y = (aA - bB) + (aB + bA)i.$$

Therefore $\mathbb{M}_2$ is isomorphic to the complex field $\mathbb{C}$.

The higher-dimensional members retain the scalar identity and complex subalgebra but lose associativity.

6 Square-root closure

Although the one-dimensional real subalgebra is not closed under square-root extraction, because -1 has no real square root, no further dimensional enlargement is required merely to guarantee the existence of square roots once the complex direction i is present.

For

$$x = a_0 + a_1 e_1 + \cdots + a_m e_m, \qquad m = n - 1,$$

define the tail sums

$$s_k = \sum_{r=k}^{m} a_r, \qquad s_{m+1} = 0.$$

If

$$x^2 = c_0 + c_1 e_1 + \cdots + c_m e_m,$$

then the general squaring formula may be written

$$c_0 = a_0^2 - s_1^2 + s_2^2$$

and, for $1 \le k \le m$,

$$c_k = 2a_0a_k + s_{k+1}^2 - s_{k+2}^2,$$

with $s_{m+2} = 0$. Consequently, for

$$C_k = \sum_{r=k}^{m} c_r,$$

the quadratic terms telescope:

$$\boxed{C_k = 2a_0s_k + s_{k+1}^2.}$$

Theorem 6.1 (Square-root closure)**.** *For every finite $n \ge 2$, the squaring map*

$$S_n : \mathbb{M}_n \longrightarrow \mathbb{M}_n, \qquad S_n(x) = x \star x,$$

is surjective. Equivalently, for every $y \in \mathbb{M}_n$ there exists at least one $x \in \mathbb{M}_n$ such that

$$x \star x = y.$$

Proof. Let

$$y = b_0 + b_1e_1 + \cdots + b_me_m \in \mathbb{M}_n.$$

If all non-scalar coefficients vanish, then $y = b_0$. For $b_0 \ge 0$, take

$$x = \sqrt{b_0}.$$

For $b_0 < 0$, since $n \ge 2$ and $i^2 = -1$, take

$$x = \sqrt{-b_0}\, i.$$

Now suppose that y has a non-scalar component. By passing to the smallest initial Myrion subalgebra containing y, let $m \ge 1$ be its highest occupied non-scalar level, so $b_m \ne 0$. Define

$$B_k = \sum_{r=k}^{m} b_r.$$

Choose $a_0 = t > 0$, set $s_{m+1} = 0$, and recursively define

$$\boxed{s_k(t) = \frac{B_k - s_{k+1}(t)^2}{2t}, \qquad k = m, m-1, \ldots, 1.}$$

Then

$$2ts_k + s_{k+1}^2 = B_k.$$

Hence the cumulative non-scalar coefficients of $x(t)^2$ satisfy

$$C_k = B_k.$$

Since $c_k = C_k - C_{k+1}$, every non-scalar coefficient is therefore the prescribed one:

$$c_k = b_k.$$

It remains only to match the scalar coefficient. Define

$$F(t) = t^2 - s_1(t)^2 + s_2(t)^2 - b_0.$$

The functions $s_k(t)$, and hence $F(t)$, are continuous for $t > 0$.

As $t \to \infty$, descending induction in the recursion gives

$$s_k(t) = O(t^{-1}),$$

so

$$F(t) = t^2 - b_0 + O(t^{-2}) \longrightarrow +\infty.$$

As $t \to 0^+$, a precise descending asymptotic establishes the opposite sign. Define

$$\alpha_k = 2^{m-k+1} - 1, \qquad d_m = \frac{b_m}{2}, \qquad d_k = -\frac{d_{k+1}^2}{2} \quad (1 \le k < m).$$

Then descending induction in the recursion gives

$$s_k(t) \sim d_k t^{-\alpha_k} \qquad (t \to 0^+).$$

Indeed, the assertion is immediate for $k = m$. If it holds at $k+1$, then $s_{k+1}(t)^2$ dominates the fixed value B_k, and the recursion gives the stated formula at k. Every d_k is non-zero. When $m \ge 2$, $\alpha_1 > \alpha_2$, so

$$\frac{s_2(t)^2}{s_1(t)^2} \longrightarrow 0.$$

For $m = 1$, $s_2 = 0$ and $s_1(t) = b_1/(2t)$. In either case,

$$F(t) \longrightarrow -\infty \qquad (t \to 0^+).$$

The intermediate value theorem now gives some $t_0 > 0$ with

$$F(t_0) = 0.$$

Recovering the coordinates by

$$a_k = s_k(t_0) - s_{k+1}(t_0)$$

produces $x(t_0) \in \mathbb{M}_n$ satisfying

$$x(t_0)^2 = y.$$

Thus S_n is surjective. $\square$

Square-root closure is an existence property, not an exhaustiveness property. For example,

$$i \in \mathbb{M}_2$$

has the usual complex roots

$$\pm\frac{1+i}{\sqrt{2}},$$

but after embedding in $\mathbb{M}_3$ it also has

$$\pm j$$

as square roots, since $j^2 = i$. More generally, for $p \ge 1$ in any ambient algebra containing e_{p+1},

$$e_{p+1}^2 = e_p.$$

Theorem 6.2 (One-level extension of square roots). *Let $x \neq 0$ be a finite-support Myrion whose highest non-zero basis component is $a_r e_r$, with $a_r \neq 0$. Then the highest occupied basis level of x^2 is either r or $r-1$. Consequently, if a non-zero element y has highest occupied basis level q, every finite-support square root of y has highest occupied level either q or $q+1$.*

Proof. If $r = 0$, then $x = a_0 \neq 0$ is a non-zero scalar and

$$x^2 = a_0^2$$

has highest occupied level 0. Thus the claimed conclusion holds in this case.

Now suppose $r \geq 1$. The highest coefficient of x^2 is

$$c_r = 2a_0 a_r.$$

If $a_0 \neq 0$, then $c_r \neq 0$, so the highest occupied level remains r.

If $a_0 = 0$, then $c_r = 0$. For $r \geq 2$, since there are no non-zero coefficients above a_r, the coefficient immediately below it is

$$c_{r-1} = a_r^2 > 0.$$

For $r = 1$, the square is purely scalar at its highest surviving level and

$$c_0 = -a_1^2 \neq 0.$$

Thus in every case squaring lowers the highest occupied level by exactly one when $a_0 = 0$, and does not lower it when $a_0 \neq 0$. Reversing the statement for a square root gives the result. ☐

Remark 6.3. *Every $\mathbb{M}_n$ with $n \geq 2$ therefore contains at least one square root of each of its elements, while embedding an element into $\mathbb{M}_{n+1}$ may reveal additional roots. No finite-support square root can require a basis level more than one step above the highest occupied level of the element being rooted.*

Remark 6.4. *The zero element has no non-zero finite-support square root. If $x \neq 0$ has highest occupied level r, the preceding theorem forces x^2 to have a non-zero component at level r or $r - 1$.*

Remark 6.5 (Finite-support roots of i)**.** *The roots $\pm j$ are not the only additional roots of i. The complete finite-support root set is*

$$\left\{ \pm\frac{1+i}{\sqrt{2}}, \quad \pm j, \quad \pm(j - 2i) \right\}.$$

The one-level extension theorem restricts a finite-support root of i to $\mathbb{M}_3$. Writing $x = a + bi + cj$, the equation $x^2 = i$ becomes

$$a^2 - b^2 - 2bc = 0, \qquad 2ab + c^2 = 1, \qquad 2ac = 0.$$

If $c = 0$, this gives the two complex roots. If $c \neq 0$, then $a = 0$, $c = \pm 1$, and $b = 0$ or $b = -2c$, giving the remaining four roots.

7 Path dependence and algebraic trajectories

For $n > 2$ and general inputs, a sequence $x_1, \ldots, x_m$ with $m \geq 3$ is not determined by its factor list alone: a full expression requires a binary association tree. Particular inputs, and all inputs in the associative algebra $\mathbb{M}_2$, may of course give association-independent values.

For three factors, the two possibilities are

$$(x_1 \star x_2) \star x_3 \quad \text{and} \quad x_1 \star (x_2 \star x_3).$$

For an ordered list of m factors, there are C_{m-1} syntactic complete parenthesisations, where C_{m-1} is a Catalan number. This counts bracketings rather than necessarily distinct resulting values; commutativity and other identities can make different trees coincide.

Each association tree defines an algebraic trajectory through intermediate Myrion values. This path dependence is distinct from ordinary sequential ordering. It prevents unrestricted reassociation or replacement of consecutive Myrion multiplications by one Myrion multiplication, although the resulting fixed real-linear maps can always be composed as ordinary matrices.

8 Multiplication operators and inversion

For fixed $x \in \mathbb{M}_n$, define the real-linear multiplication operator

$$L_x : \mathbb{M}_n \to \mathbb{M}_n, \qquad L_x(y) = x \star y.$$

Because the product is commutative, left and right multiplication coincide.

Proposition 8.1 (Unique reciprocal-solution criterion)**.** *For a fixed $\mathbb{M}_n$, the equation*

$$x \star r = 1$$

has a unique solution r if and only if

$$\det L_x \neq 0.$$

In that case,

$$\rho_n(x) := L_x^{-1}(1)$$

is called the reciprocal solution *of x in $\mathbb{M}_n$.*

Proof. The equation $x \star r = 1$ is the real-linear system $L_x r = 1$. It has exactly one solution if and only if L_x is nonsingular. $\square$

Definition 8.2 (Division by a nonsingular Myrion)**.** *For $x \notin \Sigma_n$ and $y \in \mathbb{M}_n$, define division by x in the fixed ambient algebra $\mathbb{M}_n$ by*

$$\frac{y}{x} := L_x^{-1}(y).$$

Equivalently, y/x is the unique solution $u \in \mathbb{M}_n$ of $x \star u = y$.

Remark 8.3 (Reciprocal solutions do not implement division)**.** *In a non-associative algebra, the operator L_x^{-1} need not equal $L_{\rho_n(x)}$. Thus division is generally not given by $\rho_n(x) \star y$. In $\mathbb{M}_3$, take*

$$x = 1 + i, \qquad \rho_3(x) = \frac{1-i}{2}.$$

Then

$$\rho_3(x) \star (x \star j) = \frac{1-i}{2} \star (j-1) = \frac{i+j}{2} \neq j,$$

whereas the solution of $(1+i) \star u = j$ is

$$u = \frac{1-i}{2} + j.$$

The invalid step would be to reassociate $\rho_n(x) \star (x \star y)$ as $(\rho_n(x) \star x) \star y$.

If L_x is singular, the reciprocal equation may be inconsistent or may have multiple solutions. Cancellation away from the singular division locus follows from linear injectivity: if $x \notin \Sigma_n$ and $x \star u = x \star v$, then $L_x(u - v) = 0$, hence $u = v$. No multiplication by a reciprocal solution is used.

9 The singular division locus

Definition 9.1 (Singular division locus)**.** *The singular division locus of* $\mathbb{M}_n$ *is*

$$\Sigma_n = \{x \in \mathbb{M}_n : \det L_x = 0\}.$$

For a fixed $\mathbb{M}_n$, every element of $\mathbb{M}_n \setminus \Sigma_n$ defines a unique division operation through its multiplication operator; elements of Σ_n do not define unique division. Thus Myrions are not division algebras in the standard sense for $n > 2$, but they possess *operator-defined division away from a singular division locus.* The set Σ_n is a real algebraic determinantal set rather than the "singular locus" of an algebraic variety in the usual algebraic-geometric sense.

Since the entries of L_x depend linearly on the coordinates of x, the determinant is a homogeneous polynomial of degree n. Therefore Σ_n is a homogeneous real algebraic set containing the origin.

9.1 A locus-based generalisation of the zero exclusion

For the real and complex numbers, the familiar statements

$$x \neq 0 \Longrightarrow x^{-1} \text{ exists uniquely}$$

and

$$x \neq 0 \Longrightarrow x \text{ is not a zero divisor}$$

use the isolated value 0 as the exceptional set.

The singular division locus suggests a direct higher-dimensional analogue: replace the exclusion of the single value 0 by exclusion of Σ_n.

Proposition 9.2 (Division outside the singular division locus)**.** *For* $x \in \mathbb{M}_n$*,*

$$x \notin \Sigma_n \quad \Longleftrightarrow \quad L_x \text{ is invertible.}$$

Consequently, for every $y \in \mathbb{M}_n$*, the equation* $x \star u = y$ *has a unique solution* $u = L_x^{-1}(y)$*.*

Proof. By definition,

$$x \notin \Sigma_n \quad \Longleftrightarrow \quad \det L_x \neq 0.$$

For a finite-dimensional real-linear map this is equivalent to invertibility of L_x, which gives the stated unique solution for every right-hand side y. □

Proposition 9.3 (Zero divisors and the singular division locus)**.** *For* $x \in \mathbb{M}_n$*,*

$$x \in \Sigma_n \quad \Longleftrightarrow \quad \exists\, y \neq 0 \text{ such that } x \star y = 0.$$

Hence

$$\Sigma_n = \{0\} \cup \{\text{non-zero zero divisors of } \mathbb{M}_n\},$$

and no element outside Σ_n *is a zero divisor.*

Proof. If $x \in \Sigma_n$, then L_x is singular and therefore has a non-trivial kernel. Thus there exists $y \neq 0$ with

$$L_x(y) = x \star y = 0.$$

Conversely, if $x \star y = 0$ for some $y \neq 0$, then y is a non-zero vector in the kernel of L_x, so L_x is singular and $x \in \Sigma_n$. □

For every $n > 2$, non-zero zero divisors occur explicitly. Since $i, j \in \mathbb{M}_n$,

$$(j - i) \star i = j \star i - i \star i = -1 - (-1) = 0,$$

although $j - i \neq 0$ and $i \neq 0$. Thus $j - i \in \Sigma_n$.

These two propositions show that the ordinary complex-number rule is recovered exactly when

$$\Sigma_2 = \{0\}.$$

In this sense, the familiar exclusion of zero from division is the two-dimensional special case of a broader rule:

$$\boxed{\text{operator-defined division and cancellation are available precisely away from } \Sigma_n.}$$

This reformulation does not turn $\mathbb{M}_n$ into a division algebra under the standard definition, which requires every non-zero element to admit unique division. Nor does $\mathbb{M}_n \setminus \Sigma_n$ form a multiplicatively closed arithmetic domain. In $\mathbb{M}_3$, $j \notin \Sigma_3$ and $\rho_3(j) = -i$, but both $j \star j = i$ and $-i$ belong to Σ_3. The locus instead identifies, for each fixed divisor, when its multiplication operator is bijective.

9.2 The complex case

For $z = a + bi$,

$$L_z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \qquad \det L_z = a^2 + b^2.$$

Hence

$$\Sigma_2 = \{0\}.$$

The familiar non-invertibility of zero is therefore the two-dimensional special case of the construction based on the singular division locus.

Proposition 9.4 (Dependence on the ambient dimension)**.** *Let $x \in \mathbb{M}_k$ have scalar coefficient a_0, and regard it as an element of $\mathbb{M}_n$ for $n > k$. Then*

$$\det L_x^{(n)} = a_0^{n-k} \det L_x^{(k)}.$$

Consequently, reciprocal-solution uniqueness is a property of x in a specified ambient algebra, not of its finite coordinate expression alone.

Proof. With respect to the decomposition $\mathbb{M}_n = \mathbb{M}_k \oplus \operatorname{span}\{e_k, \dots, e_{n-1}\}$, the multiplication matrix has block form

$$L_x^{(n)} = \begin{pmatrix} L_x^{(k)} & B \\ 0 & a_0 I_{n-k} \end{pmatrix},$$

because multiplication of a higher basis vector by each non-scalar component of x returns to $\mathbb{M}_k$, while the scalar component contributes a_0 times that higher vector. Taking determinants gives the formula. □

For example, i is nonsingular in $\mathbb{M}_2$ but singular in $\mathbb{M}_3$, while j is nonsingular in $\mathbb{M}_3$ but singular in $\mathbb{M}_4$.

9.3 The three-dimensional case

For $x = a + bi + cj$,

$$L_x = \begin{pmatrix} a & -b-c & -b \\ b & a & c \\ c & 0 & a \end{pmatrix}.$$

Its determinant is

$$\Delta_3(a,b,c) = a^3 + ab^2 + 2abc - bc^2 - c^3.$$

Thus

$$\Sigma_3 = \{(a,b,c) \in \mathbb{R}^3 : a^3 + ab^2 + 2abc - bc^2 - c^3 = 0\}.$$

When $\Delta_3(a,b,c) \neq 0$, the reciprocal solution is explicitly

$$\rho_3(a+bi+cj) = \frac{a^2 + (c^2 - ab)i - acj}{a^3 + ab^2 + 2abc - bc^2 - c^3}.$$

This formula gives a reciprocal solution only; as above, it does not turn division into multiplication by that element.

Unlike the positive-definite complex determinant, Δ_3 is indefinite and can vanish at non-zero points. The singular division locus is therefore not confined to the origin.

Proposition 9.5 (Basic geometry of the singular division locus)**.** *For every $n \geq 3$, Σ_n is a contractible, path-connected real algebraic set of dimension $n-1$.*

Proof. The determinant $\Delta_n(x) = \det L_x$ is homogeneous of degree n, so $x \in \Sigma_n$ implies $tx \in \Sigma_n$ for every $t \in \mathbb{R}$. The homotopy $H(x,t) = (1-t)x$ therefore contracts Σ_n to the origin.

The polynomial Δ_n is non-zero because $\Delta_n(1) = 1$, so its real zero set has dimension at most $n-1$. On the three-dimensional subalgebra, $\Delta_n(1+cj) = 1 - c^3$: the higher basis directions contribute an identity block. At $c = 1$, the derivative with respect to c is -3, so this is a smooth hypersurface point. Hence the dimension is at least $n-1$, and therefore exactly $n-1$. $\square$

Remark 9.6. *The component structure of the punctured locus, its intersection with a sphere or projectivisation, factorisation of Δ_n, and its singular strata remain to be determined in general.*

Proposition 9.7 (The singular division locus as a critical set)**.** *The singular division locus is exactly the critical set of the squaring map $S_n(x) = x \star x$:*

$$\Sigma_n = \{x \in \mathbb{M}_n : DS_n(x) \text{ is singular}\}.$$

Proof. By bilinearity and commutativity,

$$S_n(x+h) = x \star x + 2x \star h + h \star h.$$

Thus $DS_n(x) = 2L_x$, and the two operators are singular at exactly the same points. $\square$

Proposition 9.8 (Properness of the squaring map)**.** *For every finite $n \geq 2$, the squaring map $S_n : \mathbb{M}_n \to \mathbb{M}_n$ is proper. Equivalently, the inverse image of every bounded set is bounded.*

Proof. Fix any norm on $\mathbb{M}_n$. The function

$$u \longmapsto \|u \star u\|$$

is continuous and strictly positive on the compact unit sphere: strict positivity follows from the absence of non-zero square-zero elements. It therefore has a positive minimum c_n. By bilinearity,

$$\|x \star x\| \geq c_n \|x\|^2$$

for every $x \in \mathbb{M}_n$, which proves the claim. $\square$

Consequently, every regular value of S_n has finitely many roots, and that number is locally constant on the complement of the critical-value set $S_n(\Sigma_n)$. This critical-value set should be distinguished from the critical set Σ_n itself.

10 Numerical conditioning

The determinant locates exact singularity but is not a scale-independent measure of numerical proximity to it. Define

$$s(x) = \sigma_{\min}(L_x),$$

where $\sigma_{\min}$ is the smallest singular value. Then $s(x) = 0$ exactly on Σ_n.

For $\lambda \in \mathbb{R}$,

$$s(\lambda x) = |\lambda| s(x).$$

Thus the smallest singular value is an absolute, scale-dependent sensitivity measure. It is not automatically a coefficient-space distance to Σ_n; that interpretation would require a chosen norm and a structured perturbation result.

For $x \notin \Sigma_n$, the condition number is

$$\kappa(L_x) = \frac{\sigma_{\max}(L_x)}{\sigma_{\min}(L_x)}$$

measures the sensitivity of the inverse system. Near the singular division locus, small perturbations can cause large changes in $L_x^{-1}(1)$.

For $x \notin \Sigma_n$ and $\lambda \neq 0$,

$$\kappa(L_{\lambda x}) = \kappa(L_x).$$

The condition number is therefore scale-invariant and measures relative rather than absolute conditioning.

These quantities may be more useful than $|\det L_x|$ in numerical implementations.

11 Relation to established hypercomplex computation

Myrion algebras differ structurally from the Cayley–Dickson sequence. Quaternions are associative but non-commutative; octonions are neither commutative nor associative but are alternative and normed [2]. Myrions instead retain commutativity while sacrificing associativity above dimension two. They also exist in every finite dimension rather than only dimensions 2^r.

Hypercomplex neural networks commonly use fixed multiplication tables to couple real feature channels. Complex-valued and quaternion-valued networks have been used to model structured inter-channel relationships and to reduce parameter counts in some settings [6, 4, 1]. Octonion-valued networks demonstrate that non-associative arithmetic can be implemented in trainable architectures, although many analyses decompose the operations into real-valued components [8, 7].

The present work does not claim that Myrion multiplication is superior to these alternatives. It introduces a different algebraic bias: commutative binary interaction combined with association-dependent multi-factor composition and a non-trivial singular division geometry.

12 Possible application to structured neural computation

A Myrion-valued feature can be stored as n ordinary real coefficients. For the five-dimensional case,

$$X \in \mathbb{R}^{B\times T\times D\times 5}.$$

A Myrion linear layer may be defined by

$$y_u = \sum_v W_{uv} \star x_v + b_u.$$

Because each output component is a polynomial in ordinary real coefficients, such layers are compatible with automatic differentiation.

Two consecutive Myrion multiplications cannot generally be collapsed:

$$W_2 \star (W_1 \star x) \neq (W_2 \star W_1) \star x.$$

Consequently, association depth and evaluation trees remain explicit properties of the Myrion-valued architecture. For fixed W_1 and W_2, the composite map is nevertheless the ordinary real matrix product $L_{W_2}L_{W_1}$. What generally cannot be done is to represent that composite matrix as one multiplication operator L_W for another Myrion W.

Indeed, if $L_W = L_{W_2}L_{W_1}$, evaluation at the identity forces

$$W = L_W(1) = W_2 \star W_1.$$

Thus the displayed candidate is the only possible one. In every $\mathbb{M}_n$ with $n > 2$, take $W_1 = W_2 = j$. Then $W = i$ would be necessary, but

$$L_j^2(i) = -j \qquad \text{whereas} \qquad L_i(i) = -1.$$

Hence no single Myrion multiplication operator represents this composite map.

Potential experimental uses include:

- fixed left-, right-, or balanced-associated composition modules;
- learned mixtures of alternative association trees;
- recurrent states whose transition paths depend on grouping;
- regularisation based on $\sigma_{\min}(L_W)$ or $\kappa(L_W)$;
- routing or gating based on proximity to Σ_n;
- transformer projections using Myrion-valued weights with real-valued attention scores.

These are research proposals rather than demonstrated advantages. Controlled experiments should compare Myrion models with:

1. parameter-matched real-valued networks;
2. compute-matched real-valued networks;
3. n-channel networks with unconstrained learned channel mixing;
4. complex, quaternion, or parameterised hypercomplex baselines where appropriate.

Tasks involving explicit hierarchical composition would be especially informative because non-associativity changes the value represented by alternative grouping trees.

13 Limitations and open questions

The present paper is theoretical. It does not provide benchmark results, learned models, or evidence that Myrion-valued networks outperform established alternatives.

Important open questions include:

1. What standard classification, if any, best describes the Myrion family within non-associative algebra?
2. Beyond flexibility, failure of alternativity, and failure of the Jordan identity, which recognised varieties contain or exclude the Myrion family?
3. What is the component structure and singular stratification of the punctured singular division locus?
4. How many square roots does a generic Myrion possess in its own dimension and after embedding in the next dimension, and what is the geometry of these root sets?
5. How should higher power maps be classified when alternative association trees can produce different values from four or more repeated factors?
6. Are there useful invariants, conjugations, or submultiplicative norms?
7. What initialisation maintains stable activation variance?
8. Do trained models use all components or collapse to lower-dimensional subspaces?
9. Does path dependence improve compositional generalisation?
10. Can the geometry of the singular division locus provide useful gating or routing without destabilising optimisation?
11. Can fused kernels make Myrion layers computationally competitive?

Independent mathematical verification and experimental evaluation are necessary before stronger claims are warranted.

14 Conclusion

Myrion algebras form an extensible family of closed, finite-dimensional, unital commutative real algebras whose two-dimensional member is the complex field and whose higher-dimensional members are non-associative. Their multiplication is governed by a compact descending-axis rule and can be implemented directly as structured coupling among real components.

The multiplication operator L_x provides a natural account of reciprocal solutions and division. For a fixed ambient algebra, operator-defined division by x is available precisely outside the singular division locus

$$\Sigma_n = \{x : \det L_x = 0\}.$$

For complex numbers this locus is the isolated point zero; in higher dimensions it becomes a non-trivial homogeneous algebraic set.

A further structural result is that the squaring map is surjective on every finite $\mathbb{M}_n$ with $n \geq 2$. Thus the transition from the real numbers to the complex plane is the only dimensional enlargement required merely to guarantee the existence of square roots: every higher finite Myrion algebra is already square-root closed. Enlargement can still reveal additional roots, as $j^2 = i$ shows, but any finite-support square root can occupy at most one basis level above the highest occupied level of the element being rooted.

The combination of commutative binary products, association-dependent composition, square-root closure, and the geometry of the singular division locus distinguishes Myrions from more familiar hypercomplex systems. If the conventional field-style exclusion of the single value zero is generalised to exclusion of the singular division locus, then each permitted divisor induces a bijective multiplication operator and is not a zero divisor. This does not make the permitted set multiplicatively closed, nor does it make division multiplication by a reciprocal solution. Under that locus-based formulation, non-associativity remains the principal elementary arithmetic property deliberately retained as different rather than a failure to be repaired.

This is potentially significant for computation: non-associativity allows the grouping tree of a multi-factor expression to affect the value and may therefore encode hierarchical structure directly in the arithmetic. It does not make the tree recoverable from every output. These properties justify further mathematical study and controlled experiments in structured neural computation, while the present paper makes no empirical performance claim.

Data and code availability

No datasets were used. Reference implementations and symbolic verification scripts are planned for a subsequent companion release.

Conflict of interest

The author declares no conflict of interest.

References

[1] G. Altamirano-Gomez and C. Gershenson, "Quaternion convolutional neural networks: Current advances and future directions," *Advances in Applied Clifford Algebras*, vol. 34, art. 42, 2024. doi: 10.1007/s00006-024-01350-x.

[2] J. C. Baez, "The octonions," *Bulletin of the American Mathematical Society*, vol. 39, no. 2, pp. 145–205, 2002. doi: 10.1090/S0273-0979-01-00934-X.

[3] E. Grassucci, A. Zhang, and D. Comminiello, "PHNNs: Lightweight neural networks via parameterized hypercomplex convolutions," *IEEE Transactions on Neural Networks and Learning Systems*, 2022. doi: 10.1109/TNNLS.2022.3226772.

[4] T. Parcollet, M. Morchid, and G. Linarès, "Quaternion convolutional neural networks for heterogeneous image processing," in *Proc. IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, 2019, pp. 8514–8518. doi: 10.1109/ICASSP.2019.8682495.

[5] R. D. Schafer, *An Introduction to Nonassociative Algebras*. New York, NY, USA: Academic Press, 1966.

[6] C. Trabelsi et al., "Deep complex networks," in *Proc. International Conference on Learning Representations (ICLR)*, 2018.

[7] J. Wang and X. Liu, "Global μ-stability and finite-time control of octonion-valued neural networks with unbounded delays," arXiv preprint arXiv:2003.11330, 2020.

[8] J. Wu, L. Xu, Y. Kong, L. Senhadji, and H. Shu, "Deep octonion networks," *Neurocomputing*, vol. 397, pp. 179–191, 2020. doi: 10.1016/j.neucom.2020.02.053.